

THE HOMOTOPY THEORY OF SIMPLICIAL PROPS

PHILIP HACKNEY AND MARCY ROBERTSON

ABSTRACT. The category of (colored) props is an enhancement of the category of colored operads, and thus of the category of small categories. In this paper, the second in a series on ‘higher props,’ we show that the category of all small colored simplicial props admits a cofibrantly generated model category structure. With this model structure, the forgetful functor from props to operads is a right Quillen functor.

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1. INTRODUCTION

A prop is a special kind of symmetric monoidal category which is a generalization of a symmetric multicategory. We still have sets of objects, but arrows $x \rightarrow y$ are replaced by multilinear operations which may have n -inputs and m -outputs, i.e. arrows $x \rightarrow y$ are replaced by

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multilinear maps $x_1 \otimes \cdots \otimes x_n \rightarrow y_1 \otimes \cdots \otimes y_m$. The best known props, like operads, model categories of algebras such as associative, Poisson or Lie algebras. These algebras are associated to props \mathcal{T} where operations in components $\mathcal{T}(m, 1)$ generate the entire prop. Boardman and Vogt referred to props of this form as categories of “operators of standard form” in their work [3]. May independently isolated this concept, and called props of this form “operads” [20]. More general props, however, are needed if one wants to study bialgebras. Important examples which can be modeled by props but not by operads include Frobenius bialgebras (whose category is equivalent to the category of two-dimensional topological quantum field theories), topological conformal field theories (which are algebras over the chain Segal prop), and the Lie bialgebras introduced by Drinfel’d in quantization theory.

In this paper we work with colored, or multi-sorted, props where we allow the color sets to vary. Colored props are used to describe diagrams of homomorphisms of props, homotopies between homomorphisms, modules, and so forth. In particular, considering colored props is necessary if one is to study L_∞ -deformations of morphisms or more general diagrams of algebras over a prop, module-algebras, modules over an associative algebra, and Yetter-Drinfeld and Hopf modules over bialgebras.

This paper is part two of a multistage project developing the notion of “higher prop.” Here we establish that the category of simplicial props admits a cofibrantly generated model category structure. The main technical effort of this paper is contained in Proposition 4.8 where we analyze pushouts of simplicial props with varying sets of objects. While pushouts of operads have a canonical representation in the underlying category, this is not the case for pushouts in the category of props. For this reason the proof differs significantly from the corresponding proofs in the operad setting in [7, 25]. Later papers in this series will develop combinatorial models for up-to-homotopy props, following the dendroidal approach to higher operads [6, 7, 8, 22, 23] as well as Francis and Lurie’s approach to ∞ -operads in [10, 19].

2. PROPS

In the monochrome case, a prop may be defined as a symmetric monoidal category freely generated by a single object. The intuition is that this is essentially a generalization of Lawvere theories that work in non-Cartesian contexts. A *prop* \mathcal{T} consists of the following data:

- a *set* of colors $\mathfrak{C} = \text{Col}(\mathcal{T})$,

- a set of operations $\mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m)$ for every list of colors $a_1, \dots, a_n, b_1, \dots, b_m \in \mathfrak{C}$,
- a specified element $\text{id}_c \in \mathcal{T}(c; c)$ for each $c \in \mathfrak{C}$,
- an associative *vertical composition*

$$\begin{aligned} \mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \times \mathcal{T}(\langle c_j \rangle_{j=1}^p; \langle a_i \rangle_{i=1}^n) &\rightarrow \mathcal{T}(\langle c_j \rangle_{j=1}^p; \langle b_k \rangle_{k=1}^m) \\ (f, g) &\mapsto f \circ_v g, \end{aligned}$$

- an associative *horizontal composition*

$$\begin{aligned} \mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \times \mathcal{T}(\langle a_i \rangle_{i=n+1}^{n+p}; \langle b_k \rangle_{k=m+1}^{m+q}) &\rightarrow \mathcal{T}(\langle a_i \rangle_{i=1}^{n+p}; \langle b_k \rangle_{k=1}^{m+q}) \\ (f, g) &\mapsto f \circ_h g, \end{aligned}$$

- a map $\sigma^* : \mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \rightarrow \mathcal{T}(\langle a_{\sigma(i)} \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m)$ for every element $\sigma \in \Sigma_n$, and
- a map $\tau_* : \mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \rightarrow \mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_{\tau^{-1}(k)} \rangle_{k=1}^m)$ for every $\tau \in \Sigma_m$.

These data are required to satisfy the following axioms:

- The elements id_c are identities for the *vertical* composition, i.e.

$$\begin{aligned} f \circ_v (\text{id}_{a_1} \circ_h \dots \circ_h \text{id}_{a_n}) &= f \\ (\text{id}_{b_1} \circ_h \dots \circ_h \text{id}_{b_m}) \circ_v f &= f. \end{aligned}$$

- The horizontal and vertical compositions satisfy an interchange rule

$$(f \circ_v g) \circ_h (f' \circ_v g') = (f \circ_h f') \circ_v (g \circ_h g')$$

whenever the vertical compositions on the left are well-defined.

- The vertical composition is compatible with the symmetric group actions in the sense that

$$\begin{aligned} f \circ_v (\sigma_* g) &= (\sigma^* f) \circ_v g \\ \sigma^* (f \circ_v g) &= f \circ_v (\sigma^* g) \\ \tau_* (f \circ_v g) &= (\tau_* f) \circ_v g, \end{aligned}$$

where τ and σ are permutations on the appropriate number of letters.

- Suppose that f has n inputs and m outputs and g has p inputs and q outputs. If $\sigma \in \Sigma_n$, $\bar{\sigma} \in \Sigma_p$, $\tau \in \Sigma_m$, and $\bar{\tau} \in \Sigma_q$ and we write $\sigma \times \bar{\sigma} \in \Sigma_n \times \Sigma_p \hookrightarrow \Sigma_{n+p}$ then the horizontal composition satisfies

$$\begin{aligned} (\sigma^* f) \circ_h (\bar{\sigma}^* g) &= (\sigma \times \bar{\sigma})^* (f \circ_h g) \\ (\tau_* f) \circ_h (\bar{\tau}_* g) &= (\tau \times \bar{\tau})_* (f \circ_h g). \end{aligned}$$

Furthermore, if $\sigma_{xy} \in \Sigma_{x+y}$ is the permutation whose restrictions are increasing bijections

$$\begin{aligned}\sigma_{xy} : [1, y] &\xrightarrow{\cong} [x+1, x+y] \\ \sigma_{xy} : [1+y, x+y] &\xrightarrow{\cong} [1, x]\end{aligned}$$

then

$$(\sigma_{p,n})^*(\sigma_{m,q})_*(f \circ_h g) = g \circ_h f.$$

- The maps σ^* and τ_* satisfy the interchange rule $\sigma^*\tau_* = \tau_*\sigma^*$ and are *actions*:

$$\sigma^*\bar{\sigma}^* = (\bar{\sigma}\sigma)^* \quad \tau_*\bar{\tau}_* = (\tau\bar{\tau})_*.$$

Definition 2.1. A homomorphism of props $f : \mathcal{R} \rightarrow \mathcal{T}$ consists of a set map $\text{Col}(f) : \text{Col}(\mathcal{R}) \rightarrow \text{Col}(\mathcal{T})$ and for each $n, m \geq 0$ and each input-output profile $a_1, \dots, a_n; b_1, \dots, b_m$ in $\text{Col}(\mathcal{R})$ a morphism $f : \mathcal{R}(\langle a_i \rangle_{i=1}^n; \langle b_j \rangle_{j=1}^m) \rightarrow \mathcal{T}(\langle fa_i \rangle_{i=1}^n; \langle fb_j \rangle_{j=1}^m)$ which commutes with all composition, identity, and symmetry operations. The category of props and prop homomorphisms is denoted **Prop**.

It is straightforward to generalize this definition to a *prop enriched in a symmetric monoidal category* $(\mathcal{E}, \boxtimes, I)$. In the data, one replaces the sets of operations $\mathcal{T}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m)$ with objects of a closed, symmetric monoidal category \mathcal{E} , the specified elements id_c by maps $I \rightarrow \mathcal{T}(c; c)$, and all products \times by \boxtimes . In the axioms, all equalities on elements should be expressed instead by requiring that the relevant diagrams commute. In this paper we will be primarily interested when the underlying symmetric monoidal category is the category of simplicial sets with monoidal product the Cartesian product. We will write **sProp** for the category of props enriched in simplicial sets.

Example 2.2 (Module-algebras). Let $H = (H, \mu_H, \Delta_H)$ be a (co)associative bialgebra. An *H-module-algebra* is an associative algebra (A, μ_A) together with a left *H*-module structure in such a way the multiplication map on *A* becomes an *H*-module morphism. In other words, the *module-algebra axiom*

$$x(ab) = \sum_{(x)} (x_{(1)}a)(x_{(2)}b)$$

holds for $x \in H$ and $a, b \in A$, where $\Delta_H(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the Sweedler notation for comultiplication. This algebraic structure arises often in algebraic topology [4], quantum groups [18], and Lie and Hopf algebra theory. For example, in algebraic topology, the complex

cobordism ring $\mathrm{MU}^*(X)$ of a topological space X is an S -module-algebra, where S is the Landweber-Novikov algebra of stable cobordism operations.

Example 2.3 (Segal prop). The Segal prop is a prop of infinite dimensional complex orbifolds. The space of morphisms is defined as the moduli space $P_{m,n}$ of complex Riemann surfaces bounding $m+n$ labeled nonoverlapping holomorphic holes. The surfaces should be understood as compact smooth complex curves, not necessarily connected, along with $m+n$ biholomorphic maps of the closed unit disk to the surface. The precise nonoverlapping condition is that the closed disks in the inputs (outputs) do not intersect pairwise and an input disk may intersect an output disk only along the boundary. This technicality brings in the symmetric group morphisms, including the identity, to the prop, but does not create singular Riemann surfaces by composition. The moduli space means that we consider isomorphism classes of such objects. The composition of morphisms in this prop is given by sewing the Riemann surfaces along the boundaries, using the equation $zw = 1$ in the holomorphic parameters coming from the standard one on the unit disk. The tensor product of morphisms is the disjoint union. This prop plays a crucial role in conformal field theory.

Example 2.4 (Hopf-Galois extensions). An *entwining structure* [5] is a triple (A, C, ψ) , in which $A = (A, \mu)$ is an associative algebra, $C = (C, \Delta)$ is a coassociative coalgebra, and $\psi: C \otimes A \rightarrow A \otimes C$, such that the following two entwining axioms are satisfied:

$$\begin{aligned}\psi(\mathrm{id}_C \otimes \mu) &= (\mu \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \psi)(\psi \otimes \mathrm{id}_A), \\ (\mathrm{id}_A \otimes \Delta)\psi &= (\psi \otimes \mathrm{id}_C)(C \otimes \psi)(\Delta \otimes \mathrm{id}_A).\end{aligned}$$

This structure arises in the study of *coalgebra-Galois extension* and its dual notion, *algebra-Galois coextension*. There is a 2-colored prop \mathcal{Ent} whose algebras are entwining structures.

2.1. Relationship with colored operads. Another important family of examples of colored props are simply colored operads. Recall that a colored *operad* \mathcal{O} is a structure with a color set $\mathrm{Col} \mathcal{O}$ and hom sets $\mathcal{O}(c_1, \dots, c_n; c)$ for each list of colors c_1, \dots, c_n, c , together with appropriate composition operations. The precise definition is a bit more involved than that of prop (see [1]) since the operadic composition mixes together horizontal and vertical propic compositions. We regard colored operads as a special type of prop, namely those which are completely determined by the ‘one-output part’. Let **Operad** be the category of colored operads, which we shall now refer to simply as

operads (see [21]). There is a forgetful functor

$$U : \mathbf{Prop} \rightarrow \mathbf{Operad}$$

which takes \mathcal{T} to an operad $U(\mathcal{T})$ with $\text{Col } U(\mathcal{T}) = \text{Col } \mathcal{T}$. The morphism sets are defined by

$$U(\mathcal{T})(a_1, \dots, a_n; b) = \mathcal{T}(a_1, \dots, a_n; b)$$

and operadic composition

$$\gamma : U(\mathcal{T})(\langle a_i \rangle_{i=1}^n; b) \times \prod_{i=1}^n U(\mathcal{T})(\langle c_{i,j} \rangle_{j=1}^{p_i}; a_i) \rightarrow U(\mathcal{T})(\langle \langle c_{i,j} \rangle_{j=1}^{p_i} \rangle_{i=1}^n; b)$$

is given by

$$\gamma(g, \langle f_i \rangle_{i=1}^n) = g \circ_v (f_1 \circ_h \dots \circ_h f_n).$$

In [14], we indicate how an operad generates a prop and prove the following result.

Proposition 2.5. *The forgetful functor $U : \mathbf{Prop} \rightarrow \mathbf{Operad}$ has a left adjoint $F : \mathbf{Operad} \rightarrow \mathbf{Prop}$ with $\text{Col } F(\mathcal{O}) = \text{Col } \mathcal{O}$. If \mathcal{O} is an operad then*

$$F(\mathcal{O})(a_1, \dots, a_n; b) \cong \mathcal{O}(a_1, \dots, a_n; b).$$

Consequently, $UF \cong \text{id}_{\mathbf{Operad}}$.

The obvious variant holds if one instead considers operads and props enriched in \mathbf{sSet} : there is an adjunction $F : \mathbf{sOperad} \rightleftarrows \mathbf{sProp} : U$, cf. Corollary 4.2.

2.2. Graphs and megagraphs. For our purposes, a *graph* is a collection of vertices and (directed) edges; the tail of each edge is either an *input* of the graph or is connected to a vertex, and similarly the head of each edge is either an *output* of the graph or connected to a vertex. In addition, our graphs have no cycles.

To define the underlying “graphs” of the category \mathbf{Prop} , consider the free monoid monad $\mathbb{M} : \mathbf{Set} \rightarrow \mathbf{Set}$ which takes a set S to $\mathbb{M}(S) = \coprod_{k \geq 0} S^{\times k}$. There are right and left actions of the symmetric groups on the components of $\mathbb{M}S$. More compactly we could say that there are both right and left actions of the symmetric *groupoid* $\Sigma = \coprod_{n \geq 0} \Sigma_n$ on $\mathbb{M}S$. A Σ -bimodule is a set with compatible left and right Σ -actions. We now describe an extension of the notion of graph, namely one in which edges are permitted to have multiple inputs and outputs.

Definition 2.6. A *megagraph*¹ \mathcal{X} consists of a set of objects X_0 , a set of arrows X_1 , two functions $s : X_1 \rightarrow \mathbb{M}X_0$ and $t : X_1 \rightarrow \mathbb{M}X_0$, and right and left Σ actions on X_1 . These actions should have an interchange

¹ A megagraph \mathcal{X} would be called a X_0 -colored Σ -bimodule in [11].

property $\tau \cdot (x \cdot \sigma) = (\tau \cdot x) \cdot \sigma$ and should be compatible with those on $\mathbb{M}X_0$, so $t(\tau \cdot x) = \tau \cdot t(x)$ and $s(x \cdot \sigma) = s(x) \cdot \sigma$.

A map of megagraphs $f : \mathcal{X} \rightarrow \mathcal{Y}$ is determined by maps $f_0 : X_0 \rightarrow Y_0$ and $f_1 : X_1 \rightarrow Y_1$ so $sf_1 = (\mathbb{M}f_0)s$ and $tf_1 = (\mathbb{M}f_0)t$. The collection of megagraphs determines a category which we call **Mega**. A *simplicial megagraph* will be a structure with a *discrete* simplicial set of objects X_0 , a simplicial set of arrows X_1 , along with a compatible structure of a megagraph on each $(X_1)_n$. For a map of simplicial megagraphs we require that the function f_1 be a map of simplicial sets; we denote by **sMega** the category of simplicial megagraphs.

There is a forgetful functor U from **Prop** to **Mega**, defined by

$$(U\mathcal{T})_0 = \text{Col } \mathcal{T} \quad (U\mathcal{T})_1 = \coprod_{\underline{a}, \underline{b} \in \mathbb{M}(\text{Col } \mathcal{T})} \mathcal{T}(\underline{a}, \underline{b})$$

with the induced source and target maps. In the appendix of [14], we established the following.

Theorem 2.7. *The functor $U : \mathbf{Prop} \rightarrow \mathbf{Mega}$ has a left adjoint $F : \mathbf{Mega} \rightarrow \mathbf{Prop}$.*

This theorem is useful for understanding pushouts of props, but we will also use repeatedly in section 4 the following class of free prop. The (n, m) -corolla $G_{n,m}$ is the connected graph with one vertex, n inputs and m outputs, and this graph can be naturally modeled by a megagraph \mathcal{X} by letting

$$\begin{aligned} X_0 &= \{a_1, \dots, a_n, b_1, \dots, b_m\} & s(e, e) &= (a_1, \dots, a_n) \\ X_1 &= \Sigma_n \times \Sigma_m & t(e, e) &= (b_1, \dots, b_m) \end{aligned}$$

together with the evident symmetric group actions on X_1 .

Definition 2.8. If \mathcal{X} is the megagraph described above, which models the (n, m) -corolla $G_{n,m}$, we will also write $G_{n,m}$ for the prop $F(\mathcal{X})$.

This prop $G_{n,m}$ has a single generating operation in $G_{n,m}(a_1, \dots, a_n; b_1, \dots, b_m)$.

3. A REVIEW OF VARIOUS MODEL CATEGORY STRUCTURES

One of the most basic examples of a model category structure is the so-called ‘natural’ model category structure on **Cat**, the category of all small (non-enriched) categories.

Theorem 3.1. *The category **Cat** admits a cofibrantly generated model category structure where:*

- the weak equivalences are the categorical equivalences;

- the cofibrations are the functors $F : \mathcal{C} \rightarrow \mathcal{D}$ which are injective on objects;
- the fibrations are the isofibrations, i.e. functors $F : \mathcal{C} \rightarrow \mathcal{D}$ with the property that for each object c in \mathcal{C} and each isomorphism $f : Fc \rightarrow d$ in \mathcal{D} there exists a c' in \mathcal{C} and an isomorphism $g : c \rightarrow c'$ in \mathcal{C} such that $F(g) = f$.

A proof of this theorem, along with a proof that **Cat** is a simplicial model category, may be found in [24].

The Bergner model structure on simplicial categories blends the model category structure on **Cat** together with the Kan model structure on simplicial sets. We will make heavy use of this model structure throughout this paper, so we review the main parts here. Let **sSet** be the category of simplicial sets with the standard model structure [13], and let **sCat** denote the category of *simplicial categories*, by which we mean small categories enriched in **sSet**. Given a simplicial category \mathcal{A} , we can form a genuine category $\pi_0(\mathcal{A})$ which has the same set of objects as \mathcal{A} and whose set of morphisms $\pi_0(\mathcal{A})(x, y) := [\ast, \mathcal{A}(x, y)]$ is the set of path components of the simplicial set $\mathcal{A}(x, y)$. This induces a functor $\pi_0 : \mathbf{sCat} \rightarrow \mathbf{Cat}$, with values in the category of small categories. Conditions (W2) and (F2) from Theorem 3.2 below immediately imply that π_0 is a right Quillen functor, so once we have established the model structure on **sCat** we will have the (total right derived) functor $Ho(\mathbf{sCat}) \rightarrow Ho(\mathbf{Cat})$ [15, 8.5.8(2)]. In other words, any $F : \mathcal{C} \rightarrow \mathcal{D}$ in $Ho(\mathbf{sCat})$ induces a morphism $\pi_0(\mathcal{C}) \rightarrow \pi_0(\mathcal{D})$ which is well defined up to a non-unique isomorphism. Note that this set up implies that the *essential image* of a simplicial functor $F : \mathcal{A} \rightarrow \mathcal{B}$ must be defined as the full simplicial subcategory of \mathcal{B} consisting of all objects whose image in the component category $\pi_0(\mathcal{B})$ are in the essential image of the functor $\pi_0(F)$.

Let us connect this to props. Inside every prop lies a category consisting of the operations with only one input and one output. We will denote the forgetful functor $\mathbf{sProp} \rightarrow \mathbf{sCat}$ by U_0 . This is defined by $\text{Ob } U_0(\mathcal{T}) := \text{Col}(\mathcal{T})$, $U_0(\mathcal{T})(c, d) := \mathcal{T}(c; d)$, and composition is given by \circ_v . The functor U_0 admits a left adjoint, denoted by F_0 , which takes a category \mathcal{C} to an prop $F_0\mathcal{C}$ with $\text{Col}(F_0\mathcal{C}) := \text{Ob}(\mathcal{C})$. For more information see the companion paper [14]. To cut back on notation, we will simply write

$$\pi_0(\mathcal{T}) := \pi_0(U_0(\mathcal{T}))$$

for the *underlying component category* of the simplicial prop \mathcal{T} .

3.1. The Bergner model structure on sCat. The following theorem is due to Bergner [2].

Theorem 3.2. *The category of all small simplicial categories, \mathbf{sCat} , supports a right proper, cofibrantly generated, model category structure. The weak equivalences (respectively, fibrations) are the \mathbf{sSet} -enriched functors*

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

such that:

- W1 (F1):** *for all objects x, y in \mathcal{A} , the \mathbf{sSet} -morphism $F_{x,y} : \mathcal{A}(x, y) \longrightarrow \mathcal{B}(Fx, Fy)$ is a weak equivalence (respectively, fibration) in the model structure on \mathbf{sSet} , and*
- W2 (F2):** *the induced functor $\pi_0(F) : \pi_0(\mathcal{A}) \longrightarrow \pi_0(\mathcal{B})$ is a weak equivalence (respectively, fibration) of categories.*

The notions of weak equivalence and fibration in (W2) and (F2) are those from the model structure on \mathbf{Cat} given in Theorem 3.1.

Remark 3.3. Note that if $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies condition (W1), then checking that F satisfies condition (W2) is equivalent to checking that the induced functor

$$\pi_0(F) : \pi_0(\mathcal{A}) \rightarrow \pi_0(\mathcal{B})$$

is essentially surjective.

Bergner also gives an explicit description of the generating (acyclic) cofibrations of this model structure, which we will generalize in section 4. We let \emptyset denote the empty category and $\mathcal{I} = \{x\}$ for the category which has one object and one identity arrow (viewed as a simplicial category by applying the strong monoidal functor $\mathbf{Set} \rightarrow \mathbf{sSet}$). Any simplicial set K generates a simplicial category $G_{1,1}[K]$ which has two objects, arbitrarily called a_1 and b_1 , such that $\mathrm{Hom}(a_1, b_1) = K$, $\mathrm{Hom}(a_1, a_1) \cong \mathrm{Hom}(b_1, b_1) \cong *$, and $\mathrm{Hom}(b_1, a_1) = \emptyset$.

We recall the usual sets of generating cofibrations and generating acyclic cofibrations for the cofibrantly generated model category of simplicial sets; see, for example, [15, 11.1.6]. The generating cofibrations for \mathbf{sSet} are the boundary inclusions $\partial\Delta[n] \rightarrow \Delta[n]$ for $n \geq 0$, and the generating acyclic cofibrations are the horn inclusions $\Lambda[n, k] \rightarrow \Delta[n]$ for $n \geq 1$ and $0 \leq k \leq n$. The following two propositions appear in [2].

Proposition 3.4. *A functor of simplicial categories $F : \mathcal{A} \rightarrow \mathcal{B}$ is an acyclic fibration if, and only if, F has the right lifting property (RLP) with respect to all the maps*

- $G_{1,1}[K] \hookrightarrow G_{1,1}[L]$ where $K \hookrightarrow L$ is a generating cofibration of \mathbf{sSet} , and
- the maps $\emptyset \hookrightarrow \mathcal{I}$.

We consider simplicial categories \mathcal{H} with two objects x and y , weakly contractible function complexes, and only countably many simplices in each function complex. Furthermore, we require that each such \mathcal{H} is cofibrant in the Dwyer-Kan model category structure on $\mathbf{sCat}_{\{x,y\}}$ [9]. Let \mathbf{H} denote a set of representatives of isomorphism classes of such categories.

Proposition 3.5. *A functor of simplicial categories $F : \mathcal{A} \rightarrow \mathcal{B}$ is a fibration if, and only if, F has the right lifting property (RLP) with respect to maps of the form*

- $G_{1,1}[K] \hookrightarrow G_{1,1}[L]$ where $K \hookrightarrow L$ is a generating acyclic cofibration of \mathbf{sSet} , and
- the maps $\mathcal{I} \hookrightarrow \mathcal{H}$ for $\mathcal{H} \in \mathbf{H}$ which send x to x .

3.2. Categories of props with a fixed set of colors. We now fix an object set \mathfrak{C} and consider the category of all simplicial props with a *fixed* set of colors \mathfrak{C} . We will denote this category by $\mathbf{sProp}_{\mathfrak{C}}$. This category admits a model structure analogous to the Dwyer-Kan model structure on simplicial categories with a fixed object set [9]. For props, this was established by Fresse in the monochrome case [12] and Johnson and Yau [17, 3.11] in the fixed color case.

Theorem 3.6. *The category of simplicial props with fixed color set \mathfrak{C} admits a cofibrantly generated model category structure where $f : \mathcal{R} \rightarrow \mathcal{T}$ is a weak equivalence (respectively, fibration) if for each input-output profile $a_1, \dots, a_n; b_1, \dots, b_m$ in \mathfrak{C} the map*

$$f : \mathcal{R}(\langle a_i \rangle_{i=1}^n; \langle b_j \rangle_{j=1}^m) \longrightarrow \mathcal{T}(\langle fa_i \rangle_{i=1}^n; \langle fb_j \rangle_{j=1}^m)$$

is a weak equivalence (respectively, fibration) of simplicial sets.

Given a map of sets $\alpha : \mathfrak{C} \rightarrow \mathfrak{D}$ there is an induced adjoint pair of functors

$$\alpha_! : \mathbf{sProp}_{\mathfrak{C}} \rightleftarrows \mathbf{sProp}_{\mathfrak{D}} : \alpha^*.$$

Proposition 3.7. [17, 7.5] *The adjunction $(\alpha_!, \alpha^*)$ is a Quillen pair.*

If \mathcal{T} is a prop with color set \mathfrak{D} , then the prop $\alpha^*\mathcal{T}$ has colors \mathfrak{C} and operations

$$\alpha^*\mathcal{T}(\langle a_i \rangle; \langle b_j \rangle) = \mathcal{T}(\langle \alpha(a_i) \rangle; \langle \alpha(b_j) \rangle).$$

Notice that there is a canonical map $\alpha^*\mathcal{T} \rightarrow \mathcal{T}$ with color function α , and any homomorphism $\mathcal{R} \rightarrow \mathcal{T}$ with color function α factors uniquely as $\mathcal{R} \rightarrow \alpha^*\mathcal{T} \rightarrow \mathcal{T}$.

Let \mathcal{R} be a prop with color set \mathfrak{C} and suppose that $\alpha : \mathfrak{C} \rightarrow \mathfrak{D}$ is injective. Let $\mathfrak{C}' = \mathfrak{D} \setminus \alpha(\mathfrak{C})$. If we also write \mathfrak{C}' for the free prop

generated by the set \mathfrak{C}' , then

$$\alpha_1(\mathcal{R}) \cong \mathcal{R} \amalg \mathfrak{C}'.$$

This coproduct, of course, must be taken in the category \mathbf{sProp} , which is why such a formula did not appear in [17].

4. THE MODEL STRUCTURE ON \mathbf{sProp}

We now turn to the main result of the paper, namely the model structure on the category of simplicial props.

Definition 4.1. Let \mathcal{R} and \mathcal{T} be simplicial props, and let $f : \mathcal{R} \rightarrow \mathcal{T}$ be a homomorphism. We say that f is a *weak equivalence* if

W1: for each input-output profile $a_1, \dots, a_n; b_1, \dots, b_k$ in \mathcal{R} the morphism

$$f : \mathcal{R}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \longrightarrow \mathcal{T}(\langle f(a_i) \rangle_{i=1}^n; \langle f(b_k) \rangle_{k=1}^m)$$

is a weak homotopy equivalence of simplicial sets; and

W2: the underlying functor of categories $\pi_0 f : \pi_0 \mathcal{R} \rightarrow \pi_0 \mathcal{T}$ is an equivalence of categories.

We say that the morphism f is a *fibration* if

F1: for each input-output profile $a_1, \dots, a_n; b_1, \dots, b_k$ in \mathcal{R} the morphism

$$f : \mathcal{R}(\langle a_i \rangle_{i=1}^n; \langle b_k \rangle_{k=1}^m) \longrightarrow \mathcal{T}(\langle f(a_i) \rangle_{i=1}^n; \langle f(b_k) \rangle_{k=1}^m)$$

is a Kan fibration of simplicial sets; and

F2: the underlying functor of categories $\pi_0 f : \pi_0 \mathcal{R} \rightarrow \pi_0 \mathcal{T}$ is a fibration of categories.

Main Theorem. *The category of simplicial props \mathbf{sProp} admits a cofibrantly generated model category structure with the above classes of weak equivalences and fibrations, and cofibrations those morphisms which have the left lifting property (LLP) with respect to the acyclic fibrations.*

In the case where we consider simplicial categories as simplicial props, the conditions (W1) and (W2) are precisely those conditions that give a weak equivalence of simplicial categories in the Bergner model structure. In the case where \mathcal{R} and \mathcal{T} are two simplicial props with the same set \mathfrak{C} of colors and f acts as the *identity* of \mathfrak{C} , then condition (W1) implies (W2). Thus such an f is a weak equivalence if and only if it is a weak equivalence in the Johnson-Yau model structure on $\mathbf{sProp}_{\mathfrak{C}}$ from Theorem 3.6.

We further note that there is a similar model structure on the category of simplicial colored operads, **sOperad**, which was shown by the second author [25] and independently by Cisinski and Moerdijk [7].

Corollary 4.2. *The adjunction $U : \mathbf{sProp} \rightleftarrows \mathbf{sOperad} : F$ is a Quillen adjunction.*

Proof. The forgetful functor U preserves fibrations and acyclic fibrations. \square

In order to prove that our model structure is cofibrantly generated we will provide an explicit description of the generating cofibrations and generating acyclic cofibrations. In the Bergner model structure on simplicial categories the categories $G_{1,1}[X]$ play an integral role. There is a higher analogue, which we denote by $G_{n,m}[X]$, defined via the (n, m) -corollas $G_{n,m}$. Each (n, m) -corolla determines a prop $G_{n,m}$, as described in Definition 2.8. If X is a simplicial set, then the simplicial prop $G_{n,m}[X]$ is defined by $G_{n,m} \otimes X$; in particular,

$$G_{n,m}[X](a_1, \dots, a_n; b_1, \dots, b_m) = X.$$

Remark 4.3. Notice that the props $G_{n,m}[X]$ are characterized by the property that a map $f : G_{n,m}[X] \rightarrow \mathcal{T}$ consists of a set map $\{a_1, \dots, a_n, b_1, \dots, b_m\} \rightarrow \text{Col } \mathcal{T}$ together with a simplicial set map $f : X \rightarrow \mathcal{T}((\langle fa_i \rangle_{i=1}^n; \langle fb_k \rangle_{k=1}^m))$.

Definition 4.4 (Generating cofibrations). The set I of generating cofibrations consists of the following morphisms of props:

- C1:** Given a generating cofibration $K \hookrightarrow L$ in the model structure on **sSet**, the induced morphisms of props $G_{n,m}[K] \rightarrow G_{n,m}[L]$ for each $n, m \in \mathbb{N}$.
- C2:** The **sSet**-functor $\emptyset \hookrightarrow \mathcal{I}$ viewed as a prop homomorphism via the adjunction 2.1.

Definition 4.5 (Generating acyclic cofibrations). The set J of generating acyclic cofibrations consists of the following morphisms of props:

- A1:** Given a generating acyclic cofibration $K \hookrightarrow L$ of the model structure on **sSet**, the morphisms $G_{n,m}[K] \rightarrow G_{n,m}[L]$ for each $n, m \in \mathbb{N}$.
- A2:** The **sSet**-functors $\mathcal{I} \hookrightarrow \mathcal{H}$ from Proposition 3.5, viewed as prop homomorphisms.

We see that the morphisms in (A1) are weak equivalences in the Johnson-Yau model structure on **sProp** $_{\{a_1, \dots, a_n; b_1, \dots, b_m\}}$, hence are weak equivalences in **sProp**.

4.1. The lemmas. Our method of proof is to apply Kan's recognition theorem for cofibrantly generated model categories (see [15, 11.2.1] or [16, 2.1.19]). In this section, we prove several lemmas towards this goal; in particular we classify the (acyclic) fibrations in terms of lifting properties and show that pushouts of generating acyclic cofibrations are weak equivalences.

Lemma 4.6. *A morphism of simplicial props $f : \mathcal{R} \rightarrow \mathcal{T}$ is a fibration if and only if f has the right lifting property with respect to every map in J .*

Proof. Remark 4.3 implies that a map of props $f : \mathcal{R} \rightarrow \mathcal{T}$ has the right lifting property (RLP) with respect to $j : G_{n,m}[X] \hookrightarrow G_{n,m}[Y]$ for some $n, m \in \mathbb{N}$ if and only if for each input-output profile $c_1, \dots, c_n; d_1, \dots, d_m$ in \mathcal{R} , the map of simplicial sets $f : \mathcal{R}(\langle c_i \rangle_{i=1}^n, \langle d_k \rangle_{k=1}^m) \rightarrow \mathcal{T}(\langle c_i \rangle_{i=1}^n, \langle d_k \rangle_{k=1}^m)$ has the RLP with respect to the map of simplicial sets $X \hookrightarrow Y$. Thus f satisfies (F1) if and only if f has the RLP with respect to every map j in (A1).

We will now work with the set of generating acyclic cofibrations (A2); since we will also be working with the corresponding maps of simplicial categories, for clarity we write $F_0\mathcal{I} \hookrightarrow F_0\mathcal{H}$ for the maps in (A2) for the remainder of the proof.

If f is a fibration, then $U_0(f)$ is a fibration in \mathbf{sCat} , so $U_0(f)$ has the RLP with respect to all of the functors $\mathcal{I} \hookrightarrow \mathcal{H}$ from Proposition 3.5. A commutative square

$$\begin{array}{ccc} F_0\mathcal{I} & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow \\ F_0\mathcal{H} & \longrightarrow & \mathcal{T} \end{array}$$

factors as

$$\begin{array}{ccccc} F_0\mathcal{I} & \longrightarrow & F_0U_0\mathcal{R} & \longrightarrow & \mathcal{R} \\ \downarrow & \nearrow \text{dotted} & \downarrow & & \downarrow \\ F_0\mathcal{H} & \longrightarrow & F_0U_0\mathcal{T} & \longrightarrow & \mathcal{T} \end{array}$$

where the dotted lift exists since $U_0F_0 = \text{id}_{\mathbf{sCat}}$ and $U_0(f)$ has the RLP with respect to $\mathcal{I} \hookrightarrow \mathcal{H}$. Thus f has the RLP with respect to maps in (A2). On the other hand, if f has the RLP with respect to the maps $F_0\mathcal{I} \rightarrow F_0\mathcal{H}$ in (A2) and we have a commutative square

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & U_0\mathcal{R} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & U_0\mathcal{T}, \end{array}$$

we know that we have a dotted lift in

$$\begin{array}{ccccc}
 F_0\mathcal{I} & \longrightarrow & F_0U_0\mathcal{R} & \longrightarrow & \mathcal{R} \\
 \downarrow & & \downarrow & \nearrow & \downarrow \\
 F_0\mathcal{H} & \longrightarrow & F_0U_0\mathcal{T} & \longrightarrow & \mathcal{T}.
 \end{array}$$

This map factors through $F_0U_0\mathcal{R}$, and applying U_0 we have the desired lift $\mathcal{H} = U_0F_0\mathcal{H} \rightarrow U_0F_0U_0\mathcal{R} = U_0\mathcal{R}$. Thus $U_0(f)$ has the RLP with respect to maps of the form $\mathcal{I} \rightarrow \mathcal{H}$. If, furthermore, we assume that f has the RLP with respect to maps in (A2), then, in particular, it has the RLP with respect to the maps $G_{1,1}[K] \rightarrow G_{1,1}[L]$, and so $U_0(f)$ is a fibration in \mathbf{sCat} by Proposition 3.5. Thus $\pi_0(U_0(f)) = \pi_0(f)$ is a fibration of categories, and we already knew that f satisfied (F1), so f is a fibration in \mathbf{sProp} . \square

Lemma 4.7. *A morphism $f : \mathcal{R} \rightarrow \mathcal{T}$ is an acyclic fibration if and only if f has the right lifting property with respect to every map in I .*

Proof. The proof is nearly identical to that of Lemma 4.6. \square

Recall that the subcategory of *relative J -cell complexes* consists of those maps which can be constructed as a transfinite composition of pushouts of elements of J .

Proposition 4.8. *Every relative J -cell complex is a weak equivalence.*

It is, of course, enough to show that a pushout of a map in J is a weak equivalence. The proof of this proposition requires a good understanding of pushouts of simplicial props. This is the main part of our proof which differs from proofs of model structures of operads, simplicial categories, and props with fixed objects. While pushouts of operads have a canonical representation in the underlying category, this is not the case for pushouts in the category of props. For this reason the current proof now diverges from those in [7, 25].

Proposition 4.9. *Let $f : \mathcal{B} \rightarrow \mathcal{R}$ be a map of props with color map α . Then there is a unique map $\tilde{f} : \alpha_!\mathcal{B} \rightarrow \mathcal{R}$ so that $\mathcal{B} \rightarrow \mathcal{R}$ factors as $\mathcal{B} \rightarrow \alpha_!\mathcal{B} \rightarrow \mathcal{R}$. This decomposition is functorial.*

Proof. Recall that $\alpha^*\mathcal{R}(\langle a \rangle; \langle b \rangle) = \mathcal{R}(\langle sa \rangle; \langle sb \rangle)$ and that the map $\mathcal{B} \rightarrow \alpha_!\mathcal{B}$ is defined by

$$\mathcal{B} \rightarrow \alpha^*\alpha_!\mathcal{B} \rightarrow \alpha_!\mathcal{B}.$$

We thus notice that such an \tilde{f} sits inside of a diagram

$$\begin{array}{ccccc} \mathcal{B} & \longrightarrow & \alpha^* \alpha_! \mathcal{B} & \longrightarrow & \alpha_! \mathcal{B} \\ & & \downarrow \alpha^* \tilde{f} & & \downarrow \tilde{f} \\ & & \alpha^* \mathcal{R} & \longrightarrow & \mathcal{R} \end{array}$$

so we see that \tilde{f} determines and is determined by its adjoint $\tilde{f}^\vee : \mathcal{B} \rightarrow \alpha^* \mathcal{R}$.

We define the adjoint of \tilde{f}

$$\tilde{f}^\vee : \mathcal{B}(\langle a \rangle ; \langle b \rangle) \rightarrow \alpha^* \mathcal{R}(\langle a \rangle ; \langle b \rangle) = \mathcal{R}(\langle sa \rangle ; \langle sb \rangle)$$

to just be f . This is unique since we want

$$\mathcal{B} \xrightarrow{\tilde{f}^\vee} \alpha^* \mathcal{R} \longrightarrow \mathcal{R}$$

to be equal to the original map f . □

Proposition 4.10. *Let $j : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in $\mathbf{sProp}_{\mathfrak{C}}$ and $\alpha : \mathfrak{C} \rightarrow \mathfrak{D}$ a map of sets. Then*

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \alpha_! \mathcal{A} \\ \downarrow j & & \downarrow \alpha_! j \\ \mathcal{B} & \longrightarrow & \alpha_! \mathcal{B} \end{array}$$

is a pushout in \mathbf{sProp} .

Proof. Suppose we have a diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \alpha_! \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \alpha_! \mathcal{B} \end{array} \quad \begin{array}{c} \nearrow \beta \\ \searrow \gamma \\ \dashrightarrow \delta \end{array} \quad \mathcal{R}$$

where $\alpha_! \mathcal{A} \rightarrow \mathcal{R}$ has color map β . If $\beta = \text{id}_{\mathfrak{D}}$, then the dotted arrow exists and is unique by the previous proposition.

If β is not the identity, then $\mathcal{B} \rightarrow \mathcal{R}$ has color map $\beta\alpha$. Consider the diagram

$$\begin{array}{ccccc}
 \mathcal{A} & \longrightarrow & \alpha_! \mathcal{A} & \longrightarrow & \beta_! \alpha_! \mathcal{A} \\
 \downarrow j & & \downarrow \alpha_! j & & \downarrow \\
 \mathcal{B} & \xrightarrow{i} & \alpha_! \mathcal{B} & \xrightarrow{i'} & (\beta\alpha)_! \mathcal{B} \\
 & \searrow & \searrow \text{dotted} & \searrow \text{dotted} & \searrow \text{dotted} \\
 & & & & \mathcal{R}
 \end{array}$$

(Note: In the original image, there is a curved arrow from \mathcal{B} to \mathcal{R} and a dotted arrow from $(\beta\alpha)_! \mathcal{B}$ to \mathcal{R} labeled q .)

The arrow $(\beta\alpha)_! \mathcal{B} \rightarrow \mathcal{R}$ exists by the previous paragraph, which implies that the dotted arrow q exists as well. If we had some other map, say q_0 , which made the diagram

$$\begin{array}{ccc}
 \mathcal{A} & \longrightarrow & \alpha_! \mathcal{A} \\
 \downarrow j & & \downarrow \\
 \mathcal{B} & \xrightarrow{i} & \alpha_! \mathcal{B} \\
 & \searrow & \searrow \text{dotted} \\
 & & \mathcal{R}
 \end{array}$$

(Note: In the original image, the dotted arrow is labeled q_0 .)

commute, then we can factor q_0 uniquely as

$$\alpha_! \mathcal{B} \xrightarrow{i'} \beta_! \alpha_! \mathcal{B} \xrightarrow{\tilde{q}_0} \mathcal{R}.$$

Thus the composite $qi : \mathcal{B} \rightarrow \mathcal{R}$ is equal to $q_0 i = \tilde{q}_0 i' i$. Of course $qi : \mathcal{B} \rightarrow \mathcal{R}$ factors uniquely as

$$\mathcal{B} \xrightarrow{i' i} (\beta\alpha)_! \mathcal{B} \xrightarrow{\tilde{q}_i} \mathcal{R},$$

so we see that $q_0 = \tilde{q}_i$, so $q_0 = \tilde{q}_i i' = q$.

□

Lemma 4.11. *Consider the pushout square in \mathbf{sProp}*

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{f} & \mathcal{A}' \\
 \downarrow j & & \downarrow g \\
 \mathcal{B} & \xrightarrow{k} & \mathcal{B}'
 \end{array}$$

where $j : \mathcal{A} \rightarrow \mathcal{B}$ is an acyclic cofibration in $\mathbf{sProp}_{\mathfrak{C}}$, and $f : \mathcal{A} \rightarrow \mathcal{A}'$ is a morphism of props. Then $g : \mathcal{A}' \rightarrow \mathcal{B}'$ is a weak equivalence in \mathbf{sProp} .

Proof. Write $\alpha = \text{Col}(f) : \mathfrak{C} \rightarrow \mathfrak{D}$. Since $j : \mathcal{A} \rightarrow \mathcal{B}$ is an acyclic cofibration in $\mathbf{sProp}_{\mathfrak{C}}$, $\alpha_! j$ is an acyclic cofibration in $\mathbf{sProp}_{\mathfrak{D}}$ by Proposition 3.7. We know that the left hand square in

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \alpha_! \mathcal{A} & \longrightarrow & \mathcal{A}' \\ \downarrow j & & \downarrow \alpha_! j & & \downarrow g \\ \mathcal{B} & \longrightarrow & \alpha_! \mathcal{B} & \longrightarrow & \mathcal{B}' \end{array}$$

is a pushout by Proposition 4.10 and the full rectangle is a pushout by assumption, so the right rectangle is a pushout in \mathbf{sProp} as well. This implies that the right hand rectangle is a pushout in $\mathbf{sProp}_{\mathfrak{D}}$, proving that g is an acyclic cofibration in $\mathbf{sProp}_{\mathfrak{D}}$. Since g is a weak equivalence in $\mathbf{sProp}_{\mathfrak{D}}$, g is also a weak equivalence in \mathbf{sProp} . \square

A key part of the next proposition is the construction of a special filtration on the morphism space of a particular prop. To formalize this, we will exploit, in an interesting way, the infinite symmetric product² $\coprod_{n \geq 0} (S^{\times n})_{\Sigma_n}$, whose points are unordered lists, and its usual filtration

$$(1) \quad SP^k(S) = \coprod_{n=0}^k (S^{\times n})_{\Sigma_n}.$$

To see how the infinite symmetric product might come up, suppose that \mathcal{X} is a (simplicial) megagraph and $F(\mathcal{X})$ is the free (simplicial) prop on \mathcal{X} from Theorem 2.7. The construction of $F\mathcal{X}$ in [14] implies that there is a map

$$(2) \quad \text{Mor}(F\mathcal{X}) \rightarrow SP((X_1)_{\Sigma \times \Sigma}).$$

To see this, recall from [14] that an element in $\text{Mor}(F\mathcal{X})$ is an equivalence class of decorated graphs whose vertices are decorated by elements in X_1 . Then the map (2) forgets the graph entirely and puts (the orbit classes of) the elements which decorate that graph into an unordered list.

Proposition 4.12. *Let \mathcal{T} be a prop, $f : \mathcal{I} \rightarrow \mathcal{T}$ a homomorphism of props, and $j : \mathcal{I} \rightarrow \mathcal{H}$ a generating acyclic cofibration. Given the pushout diagram*

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow g \\ \mathcal{H} & \longrightarrow & \mathcal{P} \end{array}$$

the map g satisfies (F1). In particular, $\pi_0(g)$ is fully faithful.

²free abelian monoid

Proof. If S is a set we will also write S for the free prop generated by S , i.e. if \mathcal{X} is the megagraph with $X_0 = S$ and $X_1 = \emptyset$, then $S = F(\mathcal{X})$. Recall that $\text{Col}(\mathcal{H}) = \{x, y\}$ and $\text{Col}(\mathcal{I}) = \{x\}$. Let $\mathfrak{C} = \text{Col}(\mathcal{T}) \setminus \{x_*\}$, where x_* is the image of x under $f : \mathcal{I} \rightarrow \mathcal{T}$. We use the shorthand

$$\begin{aligned}\tilde{\mathcal{I}} &= \mathcal{I} \sqcup \mathfrak{C} \\ \tilde{\mathcal{H}} &= \mathcal{H} \sqcup \mathfrak{C} \\ \tilde{\mathcal{T}} &= \mathcal{T} \sqcup \{y\},\end{aligned}$$

and construct two new pushouts

$$\mathcal{P}' = \tilde{\mathcal{H}} \sqcup_{\tilde{\mathcal{I}}} \mathcal{T} \quad \text{and} \quad \mathcal{P}'' = \tilde{\mathcal{H}} \sqcup_{\tilde{\mathcal{I}} \sqcup \{y\}} \tilde{\mathcal{T}},$$

both of which are isomorphic to \mathcal{P} . To see this, notice that in the diagram

$$\begin{array}{ccccc} \mathcal{I} & \longrightarrow & \tilde{\mathcal{I}} & \longrightarrow & \mathcal{T} \\ \downarrow & & \downarrow & \text{(a)} & \downarrow \\ & & \tilde{\mathcal{I}} \sqcup \{y\} & \longrightarrow & \tilde{\mathcal{T}} \\ & & \downarrow & \text{(b)} & \downarrow \\ \mathcal{H} & \longrightarrow & \tilde{\mathcal{H}} & \longrightarrow & \mathcal{P} \end{array}$$

the left rectangle, (a), and the large square are all pushouts. Hence (a) + (b) is a pushout as well, so $\mathcal{P}' \cong \mathcal{P}$. Since (a) and (a) + (b) are both pushouts, (b) must also be a pushout, thus $\mathcal{P}'' \cong \mathcal{P}$.

We now must construct a filtration on the morphisms of the pushout \mathcal{P} , denoted by $\text{Mor}(\mathcal{P})$. First we will apply the forgetful functor $U : \mathbf{sProp} \rightarrow \mathbf{sMega}$. Given the map of megagraphs

$$\mathcal{Y} = U\mathcal{H} \sqcup_{U\mathcal{I}} U\mathcal{T} \rightarrow U\mathcal{P},$$

one observes that the composite

$$F(\mathcal{Y}) \rightarrow FU(\mathcal{P}) \rightarrow \mathcal{P}$$

is surjective on morphism spaces. The maps $(U\mathcal{H})_1 = \text{Mor } \mathcal{H} \rightarrow Y_1$ and $(U\mathcal{T})_1 = \text{Mor } \mathcal{T} \rightarrow Y_1$ induce a surjective map

$$SP(((U\mathcal{H})_1)_{\Sigma \times \Sigma}) \times SP(((U\mathcal{T})_1)_{\Sigma \times \Sigma}) \rightarrow SP((Y_1)_{\Sigma \times \Sigma}).$$

We now use the standard filtration on $SP(((U\mathcal{H})_1)_{\Sigma \times \Sigma})$ from (1) to define a filtration on $\text{Mor}(F\mathcal{Y})$, and hence one on $\text{Mor } \mathcal{P}$. An element in filtration degree k can be written, using permutations and horizontal and vertical compositions, using morphisms from \mathcal{T} and at most k nontrivial morphisms from \mathcal{H} . In light of the alternate descriptions of

\mathcal{P} above (as \mathcal{P}' and \mathcal{P}''), we could equivalently say that we have at most k nontrivial morphisms coming from $\tilde{\mathcal{H}}$.

We use the notation \mathcal{P}^k to denote the k^{th} filtration of $\text{Mor } \mathcal{P}$. The individual \mathcal{P}^k (aside from $k = 0$ and $k = \infty$) are not themselves morphism spaces for a subprop of \mathcal{P} , since \circ_h and \circ_v act additively on filtration degrees. We have

$$\text{Mor } \mathcal{T} \subset \text{Mor } \tilde{\mathcal{T}} = \mathcal{P}^0 \subset \dots \subset \mathcal{P}^k \subset \mathcal{P}^{k+1} \subset \dots \subset \text{Mor } \mathcal{P}$$

and $\text{Mor } \mathcal{P}$ is the colimit of this sequence. Note that if $a_1, \dots, a_n, b_1, \dots, b_m$ are in $\text{Col } \mathcal{T}$, then $\mathcal{T}(\langle a_i \rangle; \langle b_j \rangle) = \tilde{\mathcal{T}}(\langle a_i \rangle; \langle b_j \rangle)$.³ Thus, it remains to show that $\mathcal{P}^k(\langle a_i \rangle; \langle b_j \rangle) \xrightarrow{\sim} \mathcal{P}^{k+1}(\langle a_i \rangle; \langle b_j \rangle)$ is a weak equivalence for all k .

Suppose that we have $h \in \text{Mor } \tilde{\mathcal{H}}$ and $f \in \mathcal{P}^k$. Then

$$\begin{aligned} h \circ_h f &= (h \circ_v \langle \text{id}_{c_k} \rangle) \circ_h (\langle \text{id}_{d_\ell} \rangle \circ_v f) \\ &= (h \circ_h \langle \text{id}_{d_\ell} \rangle) \circ_v (\langle \text{id}_{c_k} \rangle \circ_h f), \end{aligned}$$

where the c_k are the inputs to h and the d_ℓ are the outputs of f . Furthermore, $(\langle \text{id}_{c_k} \rangle \circ_h f)$ is still in \mathcal{P}^k . Thus we can write any element of \mathcal{P}^{k+1} as the *vertical* composition of an element of \mathcal{P}^k with an element of $\text{Mor } \tilde{\mathcal{H}}$. Consider the diagram

$$\begin{aligned} (D_1) \quad & \coprod \left(\coprod \tilde{\mathcal{H}}(\langle c_k \rangle; \langle d_\ell \rangle) \times \mathcal{P}^{k-1}(\langle b_j \rangle; \langle c_k \rangle) \times \tilde{\mathcal{H}}(\langle a_i \rangle; \langle b_j \rangle) \right) \\ & \Rightarrow \coprod \left(\begin{array}{c} \coprod \tilde{\mathcal{H}}(\langle v_m \rangle; \langle d_\ell \rangle) \times \mathcal{P}^k(\langle a_i \rangle; \langle v_m \rangle) \\ \coprod \mathcal{P}^k(\langle w_n \rangle; \langle d_\ell \rangle) \times \tilde{\mathcal{H}}(\langle a_i \rangle; \langle w_n \rangle) \\ \coprod \mathcal{P}^g(\langle z_h \rangle; \langle d_\ell \rangle) \times \mathcal{P}^{k-g}(\langle a_i \rangle; \langle z_h \rangle) \end{array} \right) \end{aligned}$$

where the inner coproducts are over the various intermediary input-output profiles and the arrows are the two possible vertical compositions. We remark that $\tilde{\mathcal{H}}$ is in the image of $F_0 : \mathbf{sCat} \rightarrow \mathbf{sProp}$, so all morphisms of $\tilde{\mathcal{H}}$ have equal numbers of inputs and outputs. The

³ If a graph G is decorated by a coproduct of megagraphs, each connected component of G only has decorations from one of the megagraphs. Using the notation of [14], suppose that \mathfrak{G} is a decoration of a graph G by the megagraph $U\mathcal{T} \sqcup (\emptyset, \{y\})$ which represents an element of $\tilde{\mathcal{T}}(\langle a_i \rangle; \langle b_j \rangle)$. Then, splitting G into components, we see that no component of G is decorated by $(\emptyset, \{y\})$, for otherwise we would have y as an input and an output. Thus our element was actually in $\mathcal{T}(\langle a_i \rangle; \langle b_j \rangle)$.

colimit of diagram (D₁) is $\mathcal{P}^{k+1}(\langle a_i \rangle; \langle d_\ell \rangle)$. There exists a weak equivalence between (D₁) and the diagram (D₂) which is given by

$$(D_2) \quad \coprod \left(\coprod \Sigma_d^{\times 2} \times \mathcal{P}^{k-1}(\langle b_j \rangle; \langle c_k \rangle) \times \Sigma_a^{\times 2} \right) \\ \Rightarrow \coprod \left(\begin{array}{c} \coprod \Sigma_d^{\times 2} \times \mathcal{P}^k(\langle a_i \rangle; \langle v_m \rangle) \\ \coprod \mathcal{P}^k(\langle w_n \rangle; \langle d_\ell \rangle) \times \Sigma_a^{\times 2} \\ \coprod \mathcal{P}^g(\langle z_h \rangle; \langle d_\ell \rangle) \times \mathcal{P}^{k-g}(\langle a_i \rangle; \langle z_h \rangle) \end{array} \right)$$

where a denotes the length of the list $\langle a_i \rangle$. Now, notice that colimit of the diagram (D₂) is $\mathcal{P}^k(\langle a_i \rangle; \langle d_\ell \rangle)$. Thus, we have

$$\begin{array}{ccc} \text{colim}(D_2) & \xleftarrow{\simeq} \text{hocolim}(D_2) & \xleftarrow{\simeq} \text{hocolim}(D_1) \xrightarrow{\simeq} \text{colim}(D_1) \\ \downarrow = & & \downarrow = \\ \mathcal{P}^k(\langle a_i \rangle; \langle d_\ell \rangle) & \xrightarrow{\hspace{10cm}} & \mathcal{P}^{k+1}(\langle a_i \rangle; \langle d_\ell \rangle) \end{array}$$

where $\text{hocolim} \rightarrow \text{colim}$ is a weak equivalence because everything is cofibrant. Therefore, we get $\mathcal{P}^k(\langle a_i \rangle; \langle d_\ell \rangle) \simeq \mathcal{P}^{k+1}(\langle a_i \rangle; \langle d_\ell \rangle)$, as desired.

Finally, we have

$$\mathcal{T}(\langle a_i \rangle; \langle b_j \rangle) = \widetilde{\mathcal{T}}(\langle a_i \rangle; \langle b_j \rangle) = \mathcal{P}^0(\langle a_i \rangle; \langle b_j \rangle) \simeq \mathcal{P}(\langle a_i \rangle; \langle b_j \rangle),$$

and thus g satisfies (F1). \square

We are now in a position to show that relative J -cell complexes are weak equivalences.

Proof of Proposition 4.8. It is enough to show that the pushout of a generating acyclic cofibration is a weak equivalence. In other words, given $j : \mathcal{A} \rightarrow \mathcal{B}$ which is in either the set (A1) or the set (A2), if

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{T} \\ \downarrow j & & \downarrow g \\ \mathcal{B} & \xrightarrow{k} & \mathcal{P} \end{array}$$

is a pushout square in **sProp**, then g is a weak equivalence. First, let us assume that j is in (A1) and thus is of the form $j : G_{n,m}[K] \rightarrow G_{n,m}[L]$ where $K \rightarrow L$ is a generating acyclic cofibration of **sSet**. But if $K \rightarrow L$ is any acyclic cofibration in **sSet** then $j : G_{n,m}[K] \rightarrow G_{n,m}[L]$ is an acyclic cofibration in **sProp** _{$\{a_1, \dots, a_n, b_1, \dots, b_m\}$} , so g is a weak equivalence by Lemma 4.11.

Now we assume that j is an acyclic cofibration from set (A2) and consider the following pushout square:

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{f} & \mathcal{T} \\ \downarrow j & & \downarrow g \\ \mathcal{H} & \xrightarrow{k} & \mathcal{P}. \end{array}$$

We know that g satisfies (W1) by Proposition 4.12, and, furthermore, that $\pi_0(g)$ is a fully faithful functor. Thus, to show that g satisfies (W2), we must show that $\pi_0(g)$ is essentially surjective. But notice that

$$\mathrm{Ob} \pi_0 \mathcal{P} = \mathrm{Ob} U_0 \mathcal{P} = \mathrm{Col} \mathcal{P} = \mathrm{Col} \mathcal{T} \coprod_{x_* \sim x} \{x, y\} = (\mathrm{Ob} \pi_0 \mathcal{T}) \sqcup \{y\},$$

so we just need to show that y is isomorphic to an object in the image of $\pi_0(g)$; we will show that $y \cong \pi_0(g)x_*$. Let $a \in \mathcal{H}(x, y)_0$ and $b \in \mathcal{H}(y, x)_0$. Then since $\mathcal{H}(x, x)$ and $\mathcal{H}(y, y)$ are weakly contractible, $a \circ b$ is in the same path component as id_x and $b \circ a$ is in the same path component as id_y . Hence $k(a \circ b) = k(a) \circ k(b)$ is in the same path component as $k(\mathrm{id}_x) = \mathrm{id}_{\pi_0(g)x_*}$ and $k(b) \circ k(a)$ is in the same path component as id_y . Hence $y \cong \pi_0(g)x_*$ in $\pi_0 \mathcal{P}$. \square

4.2. Proof of main theorem. We can now prove the existence of the model category structure on the category of all small simplicial props.

Proof of Main Theorem. To prove the main theorem we will apply Kan's recognition theorem for cofibrantly generated model categories; we will indicate why the conditions of [16, 2.1.19] are satisfied.

The category **Prop** is bicomplete (see [14]), hence so is **sProp**. Weak equivalences are closed under retracts and satisfy the “2-out-3” property using the corresponding properties from **sSet** and **Cat**. The domains of I and J , namely $G_{n,m}[\partial \Delta^n]$, $G_{n,m}[\Lambda[n, k]]$, \emptyset , and \mathcal{I} , are all small (in the sense of model category theory and the small object argument). Thus (1), (2), and (3) hold. Our classification of fibrations and acyclic cofibrations, Lemmas 4.6 and 4.7, imply (5) and (6). Finally, (4) follows directly from Proposition 4.8 and the classification of fibrations given in Lemma 4.6. \square

The model structure on **sCat** from Theorem 3.2 was right proper, and we now show that the same is true for the model structure on **sProp**.

Lemma 4.13. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h} & \mathcal{R} \\ \downarrow f & & \downarrow g \\ \mathcal{B} & \xrightarrow{k} & \mathcal{T} \end{array}$$

be a pullback square in \mathbf{sProp} . Then, for any input-output profile $c_1, \dots, c_n; d_1, \dots, d_m$ in \mathcal{A} , the diagram

$$\begin{array}{ccc} \mathcal{A}(\langle c_i \rangle; \langle d_j \rangle) & \xrightarrow{h} & \mathcal{R}(\langle hc_i \rangle; \langle hd_j \rangle) \\ \downarrow f & & \downarrow g \\ \mathcal{B}(\langle fc_i \rangle; \langle fd_j \rangle) & \xrightarrow{k} & \mathcal{T}(\langle ghc_i \rangle; \langle kfd_j \rangle) \end{array}$$

is a pullback in \mathbf{sSet} .

Proof. A commutative diagram of the form

$$\begin{array}{ccccc} X & & & & \\ & \searrow & & & \\ & & \mathcal{A}(\langle c_i \rangle; \langle d_j \rangle) & \xrightarrow{h} & \mathcal{R}(\langle hc_i \rangle; \langle hd_j \rangle) \\ & \searrow & \downarrow f & & \downarrow g \\ & & \mathcal{B}(\langle fc_i \rangle; \langle fd_j \rangle) & \xrightarrow{k} & \mathcal{T}(\langle ghc_i \rangle; \langle kfd_j \rangle) \end{array}$$

gives

$$\begin{array}{ccc} G_{n,m}[X] & & \\ & \searrow p & \\ & & \begin{array}{ccc} \mathcal{A} & \xrightarrow{h} & \mathcal{R} \\ \downarrow f & & \downarrow g \\ \mathcal{B} & \xrightarrow{k} & \mathcal{T} \end{array} \end{array}$$

and so p induces the necessary map $X \rightarrow \mathcal{A}(\langle c_i \rangle; \langle d_j \rangle)$. \square

Proposition 4.14. *The model category structure on \mathbf{sProp} is right proper. In other words, every pullback of a weak equivalence along a fibration is a weak equivalence.*

Proof. Let

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{h} & \mathcal{R} \\ \downarrow f & & \downarrow g \\ \mathcal{B} & \xrightarrow{k} & \mathcal{T} \end{array}$$

be a pullback square in \mathbf{sProp} with g a weak equivalence and k a fibration.

Since U_0 is a right adjoint it preserves pullbacks, and, moreover, is a right Quillen functor so preserves fibrations as well. By inspection U_0 preserves weak equivalences. Thus $U_0(f)$ a weak equivalence in the model structure on \mathbf{sCat} , so f satisfies (W2). The previous lemma and the fact that \mathbf{sSet} is right proper [15, 13.1.13] imply that f satisfies (W1). \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA-RIVERSIDE
E-mail address: hackney@math.ucr.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, CANADA
E-mail address: mrober97@uwo.ca